2014 IEEE 8th Sensor Array and Multichannel Signal Processing Workshop (SAM)

Multidimensional ESPRIT: A Coupled Canonical Polyadic Decomposition Approach

Mikael Sørensen and Lieven De Lathauwer
KU Leuven, E.E. Dept. (ESAT) - STADIUS Center for Dynamical Systems, Signal Processing and Data Analytics, and iMinds Medical IT Department, Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium.
Group Science, Engineering and Technology, KU Leuven Kulak, E. Sabbelaan 53, 8500 Kortrijk, Belgium.
Email: {Mikael.Sorensen, Lieven.DeLathauwer}@kuleuven-kulak.be.

Abstract—**The ESPRIT method is a classical method for one-dimensional harmonic retrieval. During the past two decades it has become apparent that several applications in signal processing correspond to the less studied Multidimensional Harmonic Retrieval (MHR) problem. In order to accommodate this demand, we propose an extension of ESPRIT to MHR based on the coupled canonical polyadic decomposition. This leads to a dedicated uniqueness condition and an algebraic framework for MHR.**

I. Introduction

During the past two decades it has become clear that the Multidimensional Harmonic Retrieval (MHR) problem plays an important role in many signal processing applications. Despite their importance, the developments of uniqueness conditions and algorithms for MHR are lagging behind its practical use. To accommodate the use of MHR in signal processing we will introduce a link between MHR and the coupled Canonical Polyadic Decomposition (CPD) [13], [15]. This will lead to a dedicated uniqueness condition for MHR. Second, it will also lead to an algebraic method that can be understood as a generalization of the classical ESPRIT method for one-dimensional (1D) Harmonic Retrieval (HR) [9], [10] to MHR.

The rest of the introduction will present the notation. Sections II and III briefly review the MHR problem and the coupled CPD model, respectively. Section IV presents a Simultaneous matrix Diagonalization (SD) method for coupled CPD. Section V explains the link between the proposed SD method for coupled CPD and multidimensional ESPRIT. Section VI concludes the paper.

A. Notation

Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The transpose, k-rank[1], range, kernel and rth column vector of a matrix $\mathbf{A}$ are denoted by $\mathbf{A}^T$, $k_{\mathbf{A}}$, range $(\mathbf{A})$, ker $(\mathbf{A})$ and $\mathbf{a}_r$, respectively. Kronecker's delta function is denoted by δ_{ij} which is equal to one when $i = j$ and zero elsewhere. The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao product, defined as

$$\mathbf{A}\otimes\mathbf{B} \triangleq \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \ \mathbf{A}\odot\mathbf{B} \triangleq [\mathbf{a}_1\otimes\mathbf{b}_1 \ \ \mathbf{a}_2\otimes\mathbf{b}_2 \ \ \ldots\ ],$$

in which $(\mathbf{A})_{mn} = a_{mn}$. The outer product of N vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$ is denoted by $\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N}$, such that $\left(\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)}\right)_{i_1,i_2,\ldots,i_N} = a^{(1)}_{i_1}a^{(2)}_{i_2}\cdots a^{(N)}_{i_N}$. Given $\mathcal{X} \in \mathbb{C}^{I_1\times I_2\times\cdots\times I_N}$, $\mathrm{Vec}(\mathcal{X}) \in \mathbb{C}^{\prod_{n=1}^{N} I_n}$ denotes the column vector $\mathrm{Vec}(\mathcal{X}) = \left[x_{1,\ldots,1,1}, x_{1,\ldots,1,2}, \ldots, x_{I_1,\ldots,I_{N-1},I_N}\right]^T$. The reverse operation is $\mathrm{Unvec}(\mathrm{Vec}(\mathcal{X})) = \mathcal{X}$.

II. Multidimensional Harmonic Retrieval

It was recognized in [11] that N-dimensional HR problems can be cast into tensors $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times K}$ admitting a constrained Polyadic Decomposition (PD) given by

$$\mathcal{Y} = \sum_{r=1}^{R} \mathbf{a}^{(1)}_r \circ \cdots \circ \mathbf{a}^{(N)}_r \circ \mathbf{s}_r\,, \tag{1}$$

with factor matrices $\mathbf{A}^{(n)} = \left[\mathbf{a}^{(n)}_1, \ldots, \mathbf{a}^{(n)}_R\right] \in \mathbb{C}^{I_n\times R}$ and $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_R] \in \mathbb{C}^{K\times R}$ and in which $\mathbf{A}^{(n)}$ is Vandermonde, i.e.,

$$\mathbf{A}^{(n)} = \left[\mathbf{a}^{(n)}_1, \ldots, \mathbf{a}^{(n)}_R\right], \quad \mathbf{a}^{(n)}_r = \left[1, z_{r,n}, z^2_{r,n}, \ldots, z^{I_n-1}_{r,n}\right]^T. \tag{2}$$

The goal of MHR is to recover the generators $\{z_{r,n}\}$ from the observed data tensor $\mathcal{Y}$. Uniqueness conditions and algebraic methods applicable for MHR have been proposed (e.g. [11], [5], [8], [4], [6], [7], [12]). However, the existing approaches do not take the rich structure of the decomposition in (1) into account, yielding suboptimal results for MHR. To alleviate this problem, we present a link between MHR and the coupled CPD model, leading to an improved uniqueness condition tailored for MHR and an algebraic method that can be interpreted as ESPRIT for multidimensional data.

III. Coupled Canonical Polyadic Decomposition

We say that a collection of tensors $\mathcal{X}^{(n)} \in \mathbb{C}^{I_n\times J_n\times K}$, $n \in \{1, \ldots, N\}$, admits an R-term coupled PD if each tensor $\mathcal{X}^{(n)}$ can be written as [13]:

$$\mathcal{X}^{(n)} = \sum_{r=1}^{R} \mathbf{a}^{(n)}_r \circ \mathbf{b}^{(n)}_r \circ \mathbf{c}_r\,, \quad n \in \{1, \ldots, N\}, \tag{3}$$

with factor matrices $\mathbf{A}^{(n)} = \left[\mathbf{a}^{(n)}_1, \ldots, \mathbf{a}^{(n)}_R\right] \in \mathbb{C}^{I_n\times R}$, $\mathbf{B}^{(n)} = \left[\mathbf{b}^{(n)}_1, \ldots, \mathbf{b}^{(n)}_R\right] \in \mathbb{C}^{J_n\times R}$ and $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R] \in \mathbb{C}^{K\times R}$. The

[1] The k-rank of a matrix $\mathbf{A}$ is equal to the largest integer $k_{\mathbf{A}}$ such that every subset of $k_{\mathbf{A}}$ columns of $\mathbf{A}$ is linearly independent.

coupled PD of $\{\mathcal{X}^{(n)}\}$ given by (3) has the following matrix representation [13]:

$$\mathbf{X} = \mathbf{F}\mathbf{C}^T \in \mathbb{C}^{(\sum_{n=1}^{N} I_n J_n)\times K}, \quad (4)$$

where $\mathbf{F} = \left[\left(\mathbf{A}^{(1)} \odot \mathbf{B}^{(1)}\right)^T, \ldots, \left(\mathbf{A}^{(N)} \odot \mathbf{B}^{(N)}\right)^T\right]^T$. We define the coupled rank of $\{\mathcal{X}^{(n)}\}$ as the minimal number of coupled rank-1 tensors $\mathbf{a}_r^{(n)} \circ \mathbf{b}_r^{(n)} \circ \mathbf{c}_r$ that yield $\{\mathcal{X}^{(n)}\}$ in a linear combination. Assume that the coupled rank of $\{\mathcal{X}^{(n)}\}$ is R, then (3) will be called the coupled CPD of $\{\mathcal{X}^{(n)}\}$.

It is clear that the coupled rank-1 tensors in (3) can be arbitrarily permuted and that the vectors within the same coupled rank-1 tensor can be arbitrarily scaled provided the overall coupled rank-1 term remains the same. We say that the coupled CPD is unique when it is only subject to these trivial indeterminacies. Sufficient uniqueness conditions for the coupled CPD have been developed in [13]. For the case where the common factor matrix $\mathbf{C}$ has full column rank, the following result was obtained.

Theorem III.1. *Consider the coupled PD of $\mathcal{X}^{(n)} \in \mathbb{C}^{I_n\times J_n\times K}$, $n \in \{1,\ldots,N\}$ in (3). Define* [2]

$$\boldsymbol{E} = \begin{bmatrix} C_2\left(\boldsymbol{A}^{(1)}\right) \odot C_2\left(\boldsymbol{B}^{(1)}\right) \\ \vdots \\ C_2\left(\boldsymbol{A}^{(N)}\right) \odot C_2\left(\boldsymbol{B}^{(N)}\right) \end{bmatrix} \in \mathbb{C}^{\left(\sum_{n=1}^{N} \frac{I_n(I_n-1)J_n(J_n-1)}{4}\right)\times\left(\frac{R(R-1)}{2}\right)}. \quad (5)$$

If

$$\begin{cases} \boldsymbol{C} \text{ has full column rank,} \\ \boldsymbol{E} \text{ has full column rank,} \end{cases} \quad (6)$$

then the coupled rank of $\{\mathcal{X}^{(n)}\}$ is R and the coupled CPD of $\{\mathcal{X}^{(n)}\}$ is unique [13].

IV. SD method for Coupled CPD

In [1] a link between computing a CPD of a third-order tensor and SD was established. It has been further elaborated on in [2]. Here we extend the result to coupled CPD of third-order tensors. Consider the coupled PDs of the tensors $\mathcal{X}^{(n)} \in \mathbb{C}^{I_n\times J_n\times K}$, $n \in \{1,\ldots,N\}$, with matrix representation (4). Assume that $\mathbf{E}$ and $\mathbf{C}$ have full column rank. This in turn implies that $\mathbf{F}$ also has full column rank [13]. Let $\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^H$ denote the compact SVD of $\mathbf{X}$, where $\mathbf{U} \in \mathbb{C}^{(\sum_{n=1}^{N} I_n J_n)\times R}$, $\mathbf{V} \in \mathbb{C}^{K\times R}$ and $\Sigma \in \mathbb{C}^{R\times R}$. Since $\operatorname{range}(\mathbf{U}\Sigma) = \operatorname{range}(\mathbf{F})$ there exists a nonsingular matrix $\mathbf{G} \in \mathbb{C}^{R\times R}$ such that $\mathbf{F} = \mathbf{U}\Sigma\mathbf{G}$ which together with the relation $\mathbf{X} = \mathbf{F}\mathbf{C}^T = \mathbf{U}\Sigma\mathbf{V}^H$ implies that $\mathbf{C}^T = \mathbf{G}^{-1}\mathbf{V}^H$.

We will now explain how the SD procedure finds $\mathbf{G}$ from $\operatorname{range}(\mathbf{U}\Sigma)$. Partition $\mathbf{U}$ as follows

$$\mathbf{U} = \left[\mathbf{U}^{(1)T}, \ldots, \mathbf{U}^{(N)T}\right]^T, \quad \mathbf{U}^{(n)} \in \mathbb{C}^{I_n J_n\times R}.$$

[2]Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, then $C_2(\mathbf{A}) \in \mathbb{C}^{\frac{m(m-1)}{2}\times\frac{n(n-1)}{2}}$ denotes the compound matrix containing the determinants of all 2×2 submatrices of $\mathbf{A}$, arranged with the submatrix index sets in lexicographic order. See [3] and references therein for details on compound matrices.

Consider the bilinear mappings $\Phi^{(n)} : \mathbb{C}^{I_n\times J_n} \times \mathbb{C}^{I_n\times J_n} \to \mathbb{C}^{I_n\times I_n\times J_n\times J_n}$ defined by

$$\left(\Phi^{(n)}(\mathbf{X},\mathbf{Y})\right)_{ijkl} = x_{ik}y_{jl} + y_{ik}x_{jl} - x_{il}y_{jk} - y_{il}x_{jk}.$$

It is shown in [1] that $\Phi^{(n)}(\mathbf{X},\mathbf{X}) = \mathbf{0}$ if and only if $\mathbf{X}$ has at most rank 1.

Let $\mathbf{S}^{(n)} = \mathbf{U}^{(n)}\Sigma$ and $\widetilde{\mathbf{S}}^{(n,r)} = \operatorname{Unvec}\left(\mathbf{s}_r^{(n)}\right)$. For notational convenience, we denote $\mathbf{H} = \mathbf{G}^{-1}$. Since $\mathbf{a}_r^{(n)} \otimes \mathbf{b}_r^{(n)} = \operatorname{Vec}\left(\mathbf{b}_r^{(n)}\mathbf{a}_r^{(n)T}\right)$ and

$$\left(\mathbf{A}^{(n)} \odot \mathbf{B}^{(n)}\right)\mathbf{h}_r = \sum_{t=1}^{R}\left(\mathbf{a}_t^{(n)} \otimes \mathbf{b}_t^{(n)}\right)h_{tr}$$

we obtain

$$\mathcal{P}_{rs}^{(n)} \triangleq \Phi^{(n)}\left(\widetilde{\mathbf{S}}^{(n,r)}, \widetilde{\mathbf{S}}^{(n,s)}\right) = \sum_{t=1}^{R}\sum_{u=1}^{R} h_{tr}h_{us}\Phi^{(n)}\left(\mathbf{b}_t^{(n)}\mathbf{a}_t^{(n)T}, \mathbf{b}_u^{(n)}\mathbf{a}_u^{(n)T}\right).$$

Note that $\mathcal{P}_{rs}^{(n)} = \mathcal{P}_{sr}^{(n)}$. Define $\mathbf{p}^{(r,s,N)} \in \mathbb{C}^{(\sum_{n=1}^{N} I_n^2 J_n^2)}$ as follows

$$\mathbf{p}^{(r,s,N)} = \left[\operatorname{Vec}\left(\mathcal{P}_{rs}^{(1)}\right)^T, \ldots, \operatorname{Vec}\left(\mathcal{P}_{rs}^{(N)}\right)^T\right]^T,$$

where $\operatorname{Vec}\left(\mathcal{P}_{rs}^{(n)}\right) \in \mathbb{C}^{I_n^2 J_n^2}$. Assume for now that there exists a symmetric matrix $\mathbf{M} \in \mathbb{C}^{R\times R}$ which satisfies

$$\sum_{r=1}^{R}\sum_{s=1}^{R} m_{rs}\mathbf{p}^{(r,s,N)} = \mathbf{0}_{(\sum_{n=1}^{N} I_n^2 J_n^2)}, \quad (7)$$

then

$$\sum_{r=1}^{R}\sum_{s=1}^{R} m_{rs}\sum_{t=1}^{R}\sum_{u=1}^{R} h_{tr}h_{us}\Phi_{(t,u)}^{(\text{coupled})} = \mathbf{0}_{(\sum_{n=1}^{N} I_n^2 J_n^2)},$$

where

$$\Phi_{(t,u)}^{(\text{coupled})} = \begin{bmatrix} \operatorname{Vec}\left(\Phi^{(1)}\left(\mathbf{b}_t^{(1)}\mathbf{a}_t^{(1)T}, \mathbf{b}_u^{(1)}\mathbf{a}_u^{(1)T}\right)\right) \\ \vdots \\ \operatorname{Vec}\left(\Phi^{(N)}\left(\mathbf{b}_t^{(N)}\mathbf{a}_t^{(N)T}, \mathbf{b}_u^{(N)}\mathbf{a}_u^{(N)T}\right)\right) \end{bmatrix} \in \mathbb{C}^{(\sum_{n=1}^{N} I_n^2 J_n^2)}.$$

Since $\Phi_{(t,t)}^{(\text{coupled})} = \mathbf{0}_{(\sum_{n=1}^{N} I_n^2 J_n^2)}$, this reduces to

$$\sum_{r=1}^{R}\sum_{s=1}^{R} m_{rs}\sum_{t=1}^{R}\sum_{\substack{u=1\\ t\neq u}}^{R} h_{tr}h_{us}\Phi_{(t,u)}^{(\text{coupled})} = \mathbf{0}_{(\sum_{n=1}^{N} I_n^2 J_n^2)}.$$

Because of the symmetry of $\Phi^{(n)}$ and $\mathbf{M}$ we can reduce further to

$$\sum_{r=1}^{R}\sum_{s=1}^{R} m_{rs}\sum_{t=1}^{R}\sum_{\substack{u=1\\ t<u}}^{R} h_{tr}h_{us}\Phi_{(t,u)}^{(\text{coupled})} = \mathbf{0}_{(\sum_{n=1}^{N} I_n^2 J_n^2)}.$$

Stack the the column vectors $\Phi_{(t,u)}^{(\text{coupled})}$, $1 \leq t < u \leq R$, into the matrix $\Xi \in \mathbb{C}^{(\sum_{n=1}^{N} I_n^2 J_n^2)\times(\frac{R(R-1)}{2})}$ given by

$$\Xi = \left[\Phi_{(1,2)}^{(\text{coupled})}, \Phi_{(1,3)}^{(\text{coupled})}, \Phi_{(2,3)}^{(\text{coupled})}, \ldots, \Phi_{(R-1,R)}^{(\text{coupled})}\right].$$

It can verified that after removing the redundant row-vectors of the matrix Ξ we obtain the full column rank matrix $\mathbf{E}$ in (5). Under this assumption the coefficients

$$\lambda_{tu} \triangleq \sum_{r=1}^{R}\sum_{s=1}^{R} m_{rs}\sum_{t=1}^{R}\sum_{\substack{u=1\\ t<u}}^{R} h_{tr}h_{us} \quad (8)$$

must satisfy the relation $\lambda_{tu} = 0, t \neq u$. By putting the coefficients into the matrix $(\Lambda)_{tu} = \lambda_{tu}$, (8) can be reformulated as $\mathbf{M} = \mathbf{G}\Lambda\mathbf{G}^T$. At the end of this subsection we explain that any diagonal matrix Λ will generate a symmetric matrix $\mathbf{M}$ satisfying (7). Consequently, under the assumption that the vectors in the set $\{\Phi^{(\text{coupled})}_{(t,u)}\}_{t<u}$ are linearly independent, the set of possible $R \times R$ symmetric matrices $\mathbf{M}$ form a vector space of dimension R. Let $\{\mathbf{M}^{(r)}\}$ be a basis for this vector space, then we obtain the SD problem

$$\mathbf{M}^{(r)} = \mathbf{G}\Lambda^{(r)}\mathbf{G}^T, \quad r \in \{1, \ldots, R\}, \tag{9}$$

where $\Lambda^{(r)} \in \mathbb{C}^{R\times R}$ are diagonal matrices. To summarize, after calculating a basis for the solutions to

$$\sum_{r,s=1}^{R} m_{rs}\mathbf{p}^{(r,s,N)} = \sum_{s=1}^{R} m_{ss}\mathbf{p}^{(s,s,N)} + 2\sum_{t=1}^{R}\sum_{\substack{u=1\\u<t}}^{R} m_{tu}\mathbf{p}^{(t,u,N)} = \mathbf{0} \tag{10}$$

the problem has been converted to the SD problem (9) involving a congruence transform. Define

$$\begin{aligned}
\mathbf{P}^{(1)} &= \left[\mathbf{p}^{(1,1,N)}, \mathbf{p}^{(2,2,N)}, \ldots, \mathbf{p}^{(R,R,N)}\right],\\
\mathbf{P}^{(2)} &= \left[\mathbf{p}^{(1,2,N)}, \mathbf{p}^{(1,3,N)}, \mathbf{p}^{(2,3,N)}, \ldots, \mathbf{p}^{(R-1,R,N)}\right],\\
\mathbf{m} &= \left[m_{11}, m_{22}, \ldots, m_{RR}, m_{12}, m_{13}, \ldots, m_{R-1R}\right]^T,
\end{aligned}$$

then (10) can be written more compactly as

$$\mathbf{P}\mathbf{m} = \mathbf{0}_{(\sum_{n=1}^{N} I_n^2 J_n^2)}, \tag{11}$$

where

$$\mathbf{P} = \left[\mathbf{P}^{(1)}, 2\cdot\mathbf{P}^{(2)}\right] \in \mathbb{C}^{(\sum_{n=1}^{N} I_n^2 J_n^2)\times\frac{R(R+1)}{2}}. \tag{12}$$

The basis for the kernel of $\mathbf{P}$ can be found numerically from its SVD. Conversely, let $\Lambda \in \mathbb{C}^{R\times R}$ be an arbitrary diagonal matrix and $\mathbb{C}^{R\times R} \ni \mathbf{M} = \mathbf{G}\Lambda\mathbf{G}^T$. Then

$$\begin{aligned}
\sum_{r,s=1}^{R} m_{rs}\mathbf{p}^{(r,s,N)} &= \sum_{r,s=1}^{R} m_{rs}\sum_{t=1}^{R}\sum_{\substack{u=1\\u<t}}^{R} h_{tr}h_{us}\Phi^{(\text{coupled})}_{(t,u)}\\
&= \sum_{r,s=1}^{R}\sum_{\alpha,\beta=1}^{R}\sum_{t=1}^{R}\sum_{\substack{u=1\\u<t}}^{R} \lambda_{\alpha\beta}g_{r\alpha}g_{s\beta}h_{tr}h_{us}\Phi^{(\text{coupled})}_{(t,u)}.
\end{aligned}$$

Since $\sum_{r=1}^{R} h_{tr}g_{r\alpha} = \delta_{t\alpha}$ and $\sum_{s=1}^{R} h_{us}g_{s\beta} = \delta_{u\beta}$ we obtain

$$\begin{aligned}
\sum_{r,s=1}^{R} m_{rs}\mathbf{p}^{(r,s,N)} &= \sum_{\alpha,\beta=1}^{R}\sum_{t=1}^{R}\sum_{\substack{u=1\\u<t}}^{R} \lambda_{\alpha\beta}\delta_{t\alpha}\delta_{u\beta}\Phi^{(\text{coupled})}_{(t,u)}\\
&= \sum_{t=1}^{R}\sum_{\substack{u=1\\u<t}}^{R} \lambda_{tu}\Phi^{(\text{coupled})}_{(t,u)}.
\end{aligned} \tag{13}$$

Note that $\lambda_{tu} = 0$ if $t \neq u$ while $\Phi^{(\text{coupled})}_{(t,u)} = \mathbf{0}$ when $t = u$. Hence, we have shown that any diagonal matrix Λ generates a symmetric matrix that satisfies relation (7).

An outline of the SD procedure for computing a coupled CPD is presented as Algorithm 1.

Algorithm 1 SD procedure for coupled CPD.

Input: $\mathcal{X}^{(n)} = \sum_{r=1}^{R} \mathbf{a}_r^{(n)} \circ \mathbf{b}_r^{(n)} \circ \mathbf{c}_r, \quad n \in \{1, \ldots, N\}$.
Step 1: Estimate $\mathbf{C}$
- Build $\mathbf{X}$ given by (4).
- Compute SVD $\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^H$.
- Build $\mathbf{P}$ given by (12) from $\mathbf{U}\Sigma$.
- Determine R-dimensional basis $\{\mathbf{m}_r\}$ from $\ker(\mathbf{P})$.
- Stack $\{\mathbf{m}_r\}$ in symmetric matrices $\{\mathbf{M}^{(r)}\}$.
- Solve SD problem $\mathbf{M}^{(r)} = \mathbf{G}\Lambda^{(r)}\mathbf{G}^T$, $r \in \{1, \ldots, R\}$.
- Compute $\mathbf{C} = \mathbf{V}^*\mathbf{G}^{-T}$.

Step 2: Estimate $\{\mathbf{A}^{(n)}\}$ and $\{\mathbf{B}^{(n)}\}$
- Compute $\mathbf{Y}^{(n)}_{(1)} = \mathbf{X}^{(n)}_{(1)}\left(\mathbf{C}^T\right)^{\dagger}, \quad n \in \{1, \ldots, N\}$.
- Solve rank-1 approximation problems
$$\min_{\mathbf{a}_r^{(n)},\mathbf{b}_r^{(n)}} \left\|\mathbf{y}^{(n)}_{(1)} - \mathbf{a}_r^{(n)} \otimes \mathbf{b}_r^{(n)}\right\|_F^2, \quad r \in \{1, \ldots, R\}, n \in \{1, \ldots, N\}.$$

Output: $\{\mathbf{A}^{(n)}\}$, $\{\mathbf{B}^{(n)}\}$ and $\mathbf{C}$

V. Multidimensional ESPRIT

We are now ready to demonstrate that the SD method for coupled CPD can be interpreted as ESPRIT for MHR. For simplicity, we consider the two-dimensional (2D) HR ($N = 2$) problem with PD of the form (1) in which $\mathbf{S}$ has full column rank ($K \geq R$). As in ESPRIT, we exploit the shift-invariance structure of the Vandermonde matrices, yielding $\mathcal{X}^{(1)} \in \mathbb{C}^{(I_1-1)\times I_2\times 2\times K}$ with $x^{(1)}_{k_1,l_1,i_2,k} = \mathcal{Y}_{l_1+k_1-1,i_2,k}$ and matrix factorization

$$\mathbb{C}^{(I_1-1)I_2 2\times K} \ni \mathbf{X}^{(1)} = \left(\mathbf{B}^{(1)} \odot \mathbf{C}^{(1)}\right)\mathbf{S}^T, \tag{14}$$

where $\mathbf{B}^{(1)} = \mathbf{A}^{(1)}(1 : I_1 - 1, :) \odot \mathbf{A}^{(2)} \in \mathbb{C}^{(I_1-1)I_2\times R}$ and $\mathbf{C}^{(1)} = \mathbf{A}^{(1)}(1 : 2, :) \in \mathbb{C}^{2\times R}$. We also build $\mathcal{X}^{(2)} \in \mathbb{C}^{I_1\times(I_2-1)\times 2\times K}$ with $x^{(2)}_{i_1,k_2,l_2,k} = \mathcal{Y}_{i_1,l_2+k_2-1,k}$ and matrix factorization

$$\mathbb{C}^{I_1(I_2-1)2\times K} \ni \mathbf{X}^{(2)} = \left(\mathbf{B}^{(2)} \odot \mathbf{C}^{(2)}\right)\mathbf{S}^T, \tag{15}$$

where $\mathbf{B}^{(2)} = \mathbf{A}^{(1)} \odot \mathbf{A}^{(2)}(1 : I_2 - 1, :) \in \mathbb{C}^{I_1(I_2-1)\times R}$ and $\mathbf{C}^{(2)} = \mathbf{A}^{(2)}(1 : 2, :) \in \mathbb{C}^{2\times R}$. From (14) and (15) it is clear that the 2D HR problem can be computed via the SD method for coupled CPD applied to $\{\mathcal{X}^{(1)}, \mathcal{X}^{(2)}\}$. Similarly to existing algebraic methods for MHR (e.g. [11], [5], [8], [6], [7], [4], [12]) the proposed SD method admits a closed-form solution in the absence of noise. Exploiting the shift-invariance structure of the columns of the involved factor matrices, the generators $\{z_{r,n}\}$ can be obtained, e.g., $z_{r,n} = c^{(n)}_{2r}/c^{(n)}_{1r}$ in the exact case. For this reason Theorem III.1 also serves as a uniqueness condition for MHR. Let us briefly explain that coupled CPD leads to improved MHR uniqueness conditions. Consider first the case of randomly drawn generators. MHR algorithms relying on 1D HR methods (e.g. [4]) must fulfill the condition $I_1I_2 - \max(I_1, I_2) \geq R$. Vandermonde constrained CPD based methods (e.g. [5], [8], [6], [7], [12]) relax the bound to $I_1I_2 - \min(I_1, I_2) \geq R$. However, Theorem III.1 further relaxes the bound on R, see [14, Table I] for examples. In the deterministic setting Theorem III.1 also leads to improved results. As an example, if $\min(k_{\mathbf{A}^{(1)}}, k_{\mathbf{A}^{(2)}}) = 1$, then MHR algorithms relying on 1D HR methods do not work. Vandermonde constrained CPD based methods fail in cases where $\max(k_{\mathbf{A}^{(1)}}, k_{\mathbf{A}^{(2)}}) = 1$. This is in contrast to the coupled CPD based MHR uniqueness condition (6) in Theorem III.1 which cover such problems in a unified way.

We now illustrate the usefulness of Algorithm 1 for MHR. The parameters in (1) are fixed to $N = 2$, $I_1 = I_2 = 4$, $K = R = 13$ and $z_{r,n} = e^{i \cdot 2\pi \cdot \omega_{r,n}}$ in which $0 \leq \omega_{r,n} \leq 1$.

a) Case 1: We randomly generate $\{\omega_{r,n}\}$. Existing algebraic methods for MHR (e.g. [11], [5], [8], [6], [7], [4], [12]) do not apply. On the other hand, Algorithm 1 can be used. The price paid is an increased computational cost dominated by the determination of $\ker(\mathbf{P})$. In Figure 1 we plotted the true and estimated generators of $\mathbf{A}^{(1)}$ obtained by Algorithm 1. We observe that the true and estimated generators coincide (the same holds true for $\mathbf{A}^{(2)}$).

b) Case 2: To make it more difficult we now also set $\omega_{1,1} = \omega_{2,1}$, $\omega_{4,1} = \omega_{5,1}$, $\omega_{3,2} = \omega_{2,2}$ and $\omega_{5,2} = \omega_{6,2}$, implying that $k_{\mathbf{A}^{(1)}} = k_{\mathbf{A}^{(2)}} = 1$. In Figure 2 we plotted the true and estimated generators of $\mathbf{A}^{(1)}$ obtained by Algorithm 1. As expected, the true and estimated generators coincide (the same holds true for $\mathbf{A}^{(2)}$).

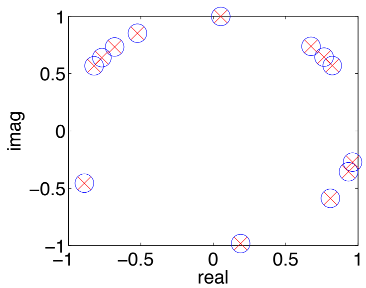

Fig. 1. True (∘) and estimated (×) generators of $\mathbf{A}^{(1)}$, case 1.

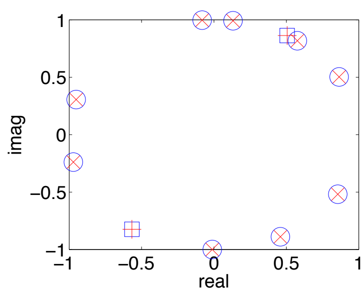

Fig. 2. True (∘) and estimated (×) isolated generators of $\mathbf{A}^{(1)}$ and true (□) and estimated (+) duplicated generators of $\mathbf{A}^{(1)}$, case 2.

VI. Conclusion

The ESPRIT method has already proven to be useful in 1D HR. However, many applications in signal processing correspond to MHR problems. This necessitates the need for the development of an algebraic framework for MHR. To accommodate this demand we introduced a link between MHR and the coupled CPD. We first briefly explained that the coupled CPD approach leads to improved uniqueness conditions for MHR. Second, we presented an algebraic SD method for coupled CPD which can be interpreted as ESPRIT for MHR. To put it differently, the coupled CPD approach does not only provide a better understanding of the MHR problem, but it also yields an algebraic ESPRIT method for MHR. More details on the link between MHR and coupled CPD and numerical experiments in the case of noisy data are provided in [14].

Acknowledgment

Research supported by: (1) Research Council KU Leuven: GOA-MaNet, CoE EF/05/006 Optimization in Engineering (OPTEC), CIF1, STRT1/08/23, (2) F.W.O.: project G.0427.10N, G.0830.14N, G.0881.14N, (3) the Belgian Federal Science Policy Office: IUAP P7 (DYSCO II, Dynamical systems, control and optimization, 2012-2017), (4) EU: ERC advanced grant no. 339804 (BIOTENSORS).

References

[1] L. De Lathauwer, "A link between the canonical decomposition in multilinear algebra and simultaneous matrix diagonalization," *SIAM J. Matrix Anal. Appl.*, vol. 28, no. 3, pp. 642–666, 2006.

[2] I. Domanov and L. De Lathauwer, "Canonical polyadic decomposition of third-order tensors: reduction to generalized eigenvalue decomposition," *Accepted for publication in SIAM J. Matrix Anal. Appl.*, available at arXiv:1312.2848.

[3] ——, "On the uniqueness of the canonical polyadic decomposition of third-order tensors — Part I: Basic results and uniqueness of one factor matrix," *SIAM J. Matrix Anal. Appl.*, vol. 34, no. 3, pp. 855–875, 2013.

[4] M. Haardt, F. Roemer, and G. Del Galdo, "Higher-order SVD-based subspace estimation to improve the parameter estimation accuracy in multidimensional harmonic retrieval problems," *IEEE Trans. Signal Process.*, vol. 56, no. 7, pp. 3198–3213, Jul. 2008.

[5] T. Jiang, N. D. Sidiropoulos, and J. M. F. Ten Berge, "Almost-sure identifiability of multidimensional harmonic retrieval," *IEEE Trans. Signal Process.*, vol. 49, no. 9, pp. 1849–1859, Sep. 2001.

[6] J. Liu and X. Liu, "An eigenvector-based approach for multidimensional frequency estimation with improved identifiability," *IEEE Trans. Signal Process.*, vol. 54, no. 12, pp. 4543–4557, Dec. 2006.

[7] J. Liu, X. Liu, and W. Ma, "Multidimensional frequency estimation with finite snapshots in the presence of identical frequencies," *IEEE Trans. Signal Process.*, vol. 55, no. 11, pp. 5179–5194, Nov. 2007.

[8] X. Liu and N. D. Sidiropoulos, "Almost sure identifiability of multidimensional constant modulus harmonic retrieval," *IEEE Trans. Signal Process.*, vol. 50, no. 9, pp. 2366–2368, Sep. 2002.

[9] A. Paulraj, R. Roy, and T. Kailath, "A subspace rotation to signal parameter estimation," *Proc. of the IEEE*, vol. 74, no. 7, pp. 1044–1045, Jul. 1986.

[10] R. Roy and T. Kailath, "ESPRIT – Estimation of signal parameters via rotational invariance techniques," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, no. 7, pp. 984–995, Jul. 1989.

[11] N. D. Sidiropoulos, "Generalizing Carathéodory's uniqueness of harmonic parameterization to N dimensions," *IEEE Trans. Inf. Theory*, vol. 47, no. 4, pp. 1687–1690, May 2001.

[12] M. Sørensen and L. De Lathauwer, "Blind signal separation via tensor decompositions with a Vandermonde factor: Canonical polyadic decomposition," *IEEE Trans. Signal Processing*, vol. 61, no. 22, pp. 5507–5519, Nov. 2013.

[13] ——, "Coupled canonical polyadic decompositions and (coupled) decompositions in multilinear rank-$(L_{r,n}, L_{r,n}, 1)$ terms — Part I: Uniqueness," ESAT-STADIUS, KU Leuven, Belgium, Tech. Rep. 13-143, 2013.

[14] ——, "Multidimensional harmonic retrieval via coupled canonical polyadic decompositions," ESAT-STADIUS, KU Leuven, Belgium, Tech. Rep. 13-240, 2013.

[15] M. Sørensen, I. Domanov, D. Nion, and L. De Lathauwer, "Coupled canonical polyadic decompositions and (coupled) decompositions in multilinear rank-$(L_{r,n}, L_{r,n}, 1)$ terms — Part II: Algorithms," ESAT-STADIUS, KU Leuven, Belgium, Tech. Rep. 13-144, 2013.